Written Exam for the M.Sc. in Economics, Winter 2010/2011

Contract Theory

Final Exam / Master's Course

January 21, 2011

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Attempt both questions

Question 1

Consider the following extension of the basic adverse selection model. A firm (the agent) interacts with a government procurement agency (the principal). The firm produces office material that the procurement agency wants to buy. The firm's cost of producing q units of office material is given by the function $C(q, \theta)$, where θ is an efficiency parameter. This function satisfies

$$C(0, \theta) = 0, \qquad C_q > 0, \qquad C_{qq} \geq 0, \qquad C_\theta > 0, \qquad C_{q\theta} > 0, \qquad C_{qq\theta} \geq 0.$$

The value for the procurement agency of receiving q units of office material is given by the function $S(q)$, which satisfies

$$S'(q) > 0, \qquad S''(q) < 0, \qquad S(0) = 0.$$

The efficiency parameter θ can take two values: $\theta \in \{\underline{\theta}, \overline{\theta}\}$, with $0 < \underline{\theta} < \overline{\theta}$. Initially (and this is where the model differs from the one we studied in the course), neither the firm nor the procurement agency knows the value of θ: they both believe that

$$\Pr[\theta = \underline{\theta}] = \nu \qquad \text{and} \qquad \Pr[\theta = \overline{\theta}] = 1 - \nu,$$

with $0 < \nu < 1$. However, the firm can, if incurring a cost $\gamma > 0$, learn the value of θ. The timing of events is as follows.

1. The procurement agency chooses a menu of contracts. A contract can specify the quantity q that the firm must produce and deliver and the payment t that the firm will receive.

2. The firm decides whether or not to incur information gathering costs γ to learn θ. The procurement agency cannot observe whether the firm incurs γ, nor can it observe the value of θ that the firm possibly learns.

3. The firm decides whether to reject all contracts in the menu or to accept one of them.

4. If the firm accepted a contract at date 3, production takes place and the procurement agency pays the firm the contractually specified payment t.

The procurement agency is risk neutral and its payoff, given a quantity q and a payment t, equals $V = S(q) - t$. The firm is also risk neutral and its payoff, given a quantity q and a payment t, equals $U = t - C(q, \theta) - \gamma$ if it has gathered information at date 2 and $U = t - C(q, \theta)$ otherwise. If the firm rejects all contracts at date 3, then its payoff equals $-\gamma$ if it has gathered information at date 2 and zero otherwise.

a) Suppose the procurement agency wants to induce the firm to gather information. Also suppose that the parameters of the model are such that it is optimal to interact with both types and to offer them distinct contracts. Then we can write the procurement agency's problem as follows. The principal chooses $\left(\underline{q}, \overline{q}, \underline{t}, \overline{t}\right)$ so as to maximize its expected payoff,

$$V\left(\underline{t}, \underline{q}, \overline{t}, \overline{q}\right) = \nu\left[S\left(\underline{q}\right) - \underline{t}\right] + (1-\nu)\left[S\left(\overline{q}\right) - \overline{t}\right],$$

subject to six constraints:

$$\overline{t} - C\left(\overline{q}, \overline{\theta}\right) \geq 0, \tag{IR-bad}$$

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq 0, \tag{IR-good}$$

$$\overline{t} - C\left(\overline{q}, \overline{\theta}\right) \geq \underline{t} - C\left(\underline{q}, \overline{\theta}\right), \tag{IC-bad}$$

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq \overline{t} - C\left(\overline{q}, \underline{\theta}\right), \tag{IC-good}$$

$$\begin{aligned} & \nu\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1-\nu)\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right] - \gamma \\ \geq\; & \underline{t} - \nu C\left(\underline{q}, \underline{\theta}\right) - (1-\nu) C\left(\underline{q}, \overline{\theta}\right), \end{aligned} \tag{IG-good}$$

$$\begin{aligned} & \nu\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1-\nu)\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right] - \gamma \\ \geq\; & \overline{t} - \nu C\left(\overline{q}, \underline{\theta}\right) - (1-\nu) C\left(\overline{q}, \overline{\theta}\right). \end{aligned} \tag{IG-bad}$$

Explain (briefly) in words what each one of the six constraints says.

b) Let the first-best quantities, $\overline{q}^{FB}$ and $\underline{q}^{FB}$, be defined in the usual way by $S'\left(\overline{q}^{FB}\right) = C_q\left(\overline{q}^{FB}, \overline{\theta}\right)$ and $S'\left(\underline{q}^{FB}\right) = C_q\left(\underline{q}^{FB}, \underline{\theta}\right)$. Let the second-best quantities, $\overline{q}^{SB}$ and $\underline{q}^{SB}$, be the ones that solve the above problem. Show (by solving as much as you need of the problem) how $\overline{q}^{SB}$ relates to $\overline{q}^{FB}$, and how $\underline{q}^{SB}$ relates to $\underline{q}^{FB}$. You are allowed to assume that the second-order condition is satisfied (and you will not get any credit if you nevertheless investigate that.)

- *Hint*: Is (IC-bad) implied by (IG-good)? Is (IC-good) implied by (IG-bad)?

Question 2

Consider the following moral hazard model with mean-variance preferences that we studied in the course. There is one (single) agent, A, and one principal, P. A chooses an effort level $e \in \Re_+$, thereby incurring the cost $c(e) = \frac{1}{2}e^2$. Given a choice of e, the output (i.e., A's performance) equals $q = e + z$, where z is an exogenous random term drawn from a normal distribution with mean zero and variance ν. It is assumed that P can observe q but not e. Moreover, neither P nor A can observe z. A's wage (i.e., the transfer from P to A) can only be contingent on the output q. It is restricted to be linear in q:

$$t = \alpha + \beta q = \alpha + \beta (e + z).$$

A is risk averse and has a CARA utility function: $U = -\exp[-r(t - c(e))]$, where r (> 0) is the coefficient of absolute risk aversion. Therefore A's expected utility is

$$EU = -\int_{-\infty}^{\infty} \exp[-r(t - c(e))] f(z) \, dz,$$

where $f(z)$ is the density of the normal distribution. P's objective function is

$$V = q - t = q - \alpha - \beta q = (1 - \beta)(e + z) - \alpha,$$

which in expected terms becomes $EV = (1 - \beta) e - \alpha$. It is also assumed that A's outside option utility is $\widehat{U} = -\exp\left[-r\widehat{t}\right]$, where $\widehat{t} > 0$. The timing of events is as follows.

1. P chooses the contract parameters, α and β.
2. A accepts or rejects the contract and, if accepting, chooses an effort level.
3. The noise term z is realized and A and P get their payoffs.

Answer the following questions:

a) Solve for the β-parameter in the second-best optimal contract, denoted β^{SB} (you do not need to solve for α^{SB}, and you will not get any credit if you nevertheless do that). You should make use of the following (well-known) result:

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

b) *[You are encouraged to attempt parts b), c) and d) even if you have not been able to answer parts a).]* Does the agent get any rents at the second-best optimum? Do not only answer yes or no, but also explain how you can tell. [PLEASE TURN THE PAGE!]

c) The first-best values of the effort level and the β-parameter equal $e^{FB} = 1$ and $\beta^{FB} = 0$, respectively. How do these values relate to the corresponding second-best values? In particular, is there under- or overprovision of effort at the second-best optimum?

d) Consider the limit case where $r \to 0$. Explain what happens to the relationship between the second-best and the first-best effort levels. Also explain the intuition for this result.

END OF EXAM